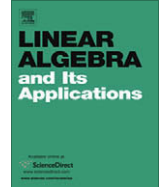


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The spectral radius of graphs without paths and cycles of specified length

Vladimir Nikiforov

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, United States

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ABSTRACT

Let G be a graph with n vertices and $\mu(G)$ be the largest eigenvalue of the adjacency matrix of G . We study how large $\mu(G)$ can be when G does not contain cycles and paths of specified order. In particular, we determine the maximum spectral radius of graphs without paths of given length, and give tight bounds on the spectral radius of graphs without given even cycles. We also raise a number of open problems.

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1. Introduction

How large can be the spectral radius $\mu(G)$ of a graph G of order n without a path or cycle of order k ? Such questions come easily into mind when one studies the problems of extremal graph theory. Thus, recall the general Turán type problem:

What is the maximum number of edges in a graph G of order n if G does not contain subgraphs of particular kind.

In [3], Brualdi and Solheid raised an analogous spectral problem:

What is $\max \mu(G)$ if the graph G belongs to a specified class of graphs.

Blending these two questions, we obtain a Brualdi–Solheid–Turán type problem:

What is $\max \mu(G)$ if G is a graph of order n and G does not contain subgraphs of particular kind.

E-mail address: vnikifrv@memphis.edu

Examples of such problems are numerous since to every Turán type problem corresponds a Brualdi–Solheid–Turán type problem. In fact, many fundamental types of graphs, like e.g. *planar* or *k-chromatic*, are characterized by forbidden graphs, so the study of Brualdi–Solheid–Turán type problems is an important topic in spectral graph theory.

In this paper we focus on the maximum spectral radius of graphs of order n without paths or cycles of specified length.

Write C_k and P_k for the cycle and path of order k , and let

$$f_l(n) = \max \{ \mu(G) : |G| = n, C_l \not\subseteq G \},$$

$$g_l(n) = \max \{ \mu(G) : |G| = n, C_l \not\subseteq G \text{ and } C_{l+1} \not\subseteq G \}$$

$$h_l(n) = \max \{ \mu(G) : |G| = n, P_l \not\subseteq G \}.$$

Perhaps it is more natural to define $g_l(n)$ as $\max \{ \mu(G) : |G| = n, \text{ and } C_p \not\subseteq G \text{ for } p \geq l \}$. However, this more relaxed definition seems to determine precisely the same function $g_l(n)$, as suggested in Conjecture 15 in the concluding section of this paper.

The value of $f_l(n)$ for odd l

For odd l the function $f_l(n)$ was essentially determined in [13]: if l is odd and $n > 321l$, then

$$f_l(n) = \sqrt{\lfloor n^2/4 \rfloor}.$$

The smallest ratio n/l for which the equation is still valid is not known. Note that the complete bipartite graph with color classes of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ has no odd cycles and its spectral radius is precisely $\sqrt{\lfloor n^2/4 \rfloor}$. Thus, for odd l we have $f_l(n) \sim n/2$. As we shall see below, this is in sharp contrast with the value of $f_l(n)$ for even l .

Bounds on $f_4(n)$

The value of $f_4(n)$ was essentially determined in [12]:

Let G be a graph of order n with $\mu(G) = \mu$. If $C_4 \not\subseteq G$, then

$$\mu^2 - \mu \leq n - 1. \quad (1)$$

Equality holds if and only if every two vertices of G have exactly one common neighbor, i.e., when G is the friendship graph.

An easy calculation implies that

$$f_4(n) = 1/2 + \sqrt{n - 3/4} + O(1/n),$$

where for odd n the $O(1/n)$ term is zero. Finding the precise value of $f_4(n)$ for even n is an open problem.

Bounds on $f_l(n)$ for even $l > 4$

The inequality (1) can be generalized for arbitrary even cycles in the following way: if $C_{2k+2} \not\subseteq G$, then

$$\mu^2 - (k-1)\mu \leq k(n-1).$$

This inequality and a matching lower bound imply that

$$(k-1)/2 + \sqrt{kn} + o(n) \leq f_{2k+2}(n) \leq k/2 + \sqrt{kn} + o(n). \quad (2)$$

The exact value of $f_{2k+2}(n)$ is not known for $k \geq 2$, and finding this value seems a challenge. Nevertheless, the precision of (2) is somewhat surprising, given that the asymptotics of the maximum number of edges in C_{2k+2} -free graphs of order n is not known for $k \geq 2$.

Bounds on $g_l(n)$

Let us consider now the function $g_l(n)$. To begin with, Favaron, Mahéo, and Saclé [8] showed that if a graph G of order n contains neither C_3 nor C_4 , then $\mu(G) \leq \sqrt{n-1}$. Since the star of order n has no cycles and its spectral radius is $\sqrt{n-1}$, we deduce that

$$g_3(n) = \sqrt{n-1}.$$

We do not know the exact value of $g_l(n)$ for $l > 3$. Nevertheless, an example, together with the inequality $g_{2k+1}(n) \leq f_{2k+2}(n)$ and relation (2), gives

$$(k-1)/2 + \sqrt{kn} + o(n) \leq g_{2k+1}(n) \leq k/2 + \sqrt{kn} + o(n);$$

thus, $g_{2k+1}(n)$ is known within an additive term not exceeding $1/2$.

Luckily, for even l we can give almost exact asymptotics of $g_l(n)$:

$$g_{2k}(n) = (k-1)/2 + \sqrt{kn} + O(n^{-1/2}).$$

Bounds on $h_l(n)$

Finally, for $h_l(n)$ we have precise results when n is sufficiently large:

$$h_{2k}(n) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4},$$

$$h_{2k+1}(n) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4} + 1/n + O(n^{-3/2}).$$

In addition, for every $l \geq 4$, we know the unique graph for which $h_l(n)$ is attained when n is sufficiently large. Specifically, $h_{2k+1}(n)$ is known exactly but cannot be given by a simple closed expression.

The main results of the paper are stated in the next section: first lower bounds, and then upper bounds on $f_{2l}(n)$, $g_l(n)$ and $h_l(n)$. The proofs of these results are given in Section 3. At the end of the paper we state two conjectures, outlining possible solutions of related problems.

2. Main results

First we recall some notation, which in general follows [2]; thus, if G is a graph, we write:

- $V(G)$ for the vertex set of G ;
- $|G|$ for the number of vertices of G ;
- $E(G)$ for the edge set of G and $e(G)$ for $|E(G)|$;
- $\delta(G)$ for the minimum degree of G ;
- $G - u$ for the graph obtained by removing the vertex $u \in V(G)$;
- $\Gamma(u)$ for the set of neighbors of a vertex u and $d(u)$ for $|\Gamma(u)|$;
- $e_G(X)$ for the number of edges induced by a set $X \subset V(G)$;
- $e_G(X, Y)$ for the number of edges joining vertices in X to vertices in Y , where X and Y are disjoint subsets of $V(G)$.

We write K_p and \overline{K}_p for the complete and the edgeless graph of order p .

2.1. Lower bounds on $f_{2l}(n)$, $g_l(n)$ and $h_l(n)$

The lower bounds on $f_{2l}(n)$, $g_l(n)$ and $h_l(n)$ are given by two families of graphs, which for sufficiently large n give the exact values of $h_l(n)$, and perhaps also of $f_{2l}(n)$ and $g_l(n)$; for a precise statement see Conjecture 15 in the concluding remarks.

Suppose that $1 \leq k < n$ and let:

- $S_{n,k}$ be the graph obtained by joining every vertex of a complete graph of order k to every vertex of an independent set of order $n - k$, that is to say $S_{n,k}$ is the join of K_k and \bar{K}_{n-k} ;
- $S_{n,k}^+$ be the graph obtained by adding one edge within the independent set of $S_{n,k}$.

Clearly, $S_{n,k}$ and $S_{n,k}^+$ are graphs of order n and

$$e(S_{n,k}) = kn - (k^2 + k)/2, \quad e(S_{n,k}^+) = kn - (k^2 + k)/2 + 1.$$

To calculate $\mu(S_{n,k})$, let $\mu = \mu(S_{n,k})$ and apply a theorem of Finck and Grohmann [7] (see also [4, Theorem 2.8]) getting

$$\mu^2 - (k-1)\mu - k(n-k) = 0.$$

Thus, we have

$$\mu(S_{n,k}) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4}. \quad (3)$$

A slightly more involved approach gives $\mu(S_{n,k}^+)$ as follows.

Proposition 1. $\mu(S_{n,k}^+)$ is the largest root of the equation

$$x^3 - kx^2 - (kn - k^2 - k + 1)x + k(n - k - 2) = 0,$$

and satisfies the inequalities

$$\frac{1}{n - k + \sqrt{kn/2}} < \mu(S_{n,k}^+) - \mu(S_{n,k}) < \frac{1}{n - k - 2\sqrt{(n-k)/k}}. \quad (4)$$

After some simple algebra, inequalities (3) and (4) give

$$\mu(S_{n,k}^+) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4} + 1/n + O(n^{-3/2}).$$

Note that $P_{l+1} \not\subseteq S_{n,k}$ and $C_l \not\subseteq S_{n,k}$ for $l \geq 2k + 1$. Likewise, $P_{l+1} \not\subseteq S_{n,k}$ and $C_l \not\subseteq S_{n,k}$ for $l \geq 2k + 2$. Therefore, we obtain the following bounds

$$h_{2k}(n) \geq \mu(S_{n,k}) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4}, \quad (5)$$

$$h_{2k+1}(n) \geq \mu(S_{n,k}^+) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4} + 1/n + O(n^{-3/2}), \quad (6)$$

$$g_{2k}(n) \geq \mu(S_{n,k}) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4}, \quad (7)$$

$$g_{2k+1}(n) \geq \mu(S_{n,k}^+) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4} + 1/n + O(n^{-3/2}), \quad (8)$$

$$f_{2k+2}(n) \geq \mu(S_{n,k}^+) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4} + 1/n + O(n^{-3/2}). \quad (9)$$

2.2. Upper bounds on $f_l(n)$, $g_l(n)$ and $h_l(n)$

Theorem 2. Let $k \geq 1$, $n \geq 2^{4k}$ and G be a graph of order n .

- If $\mu(G) \geq \mu(S_{n,k})$, then G contains a P_{2k+2} unless $G = S_{n,k}$.
- If $\mu(G) \geq \mu(S_{n,k}^+)$, then G contains a P_{2k+3} unless $G = S_{n,k}^+$.

Theorem 2 implies that for every $k \geq 1$ and $n \geq 2^{4k}$, we have

$$h_{2k}(n) = \mu(S_{n,k}) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4},$$

$$h_{2k+1}(n) = \mu(S_{n,k}^+) = (k-1)/2 + \sqrt{kn - (3k^2 + 2k - 1)/4} + 1/n + O(n^{-3/2}).$$

Theorem 3. Let $k \geq 1$ and G be a graph of order n . If

$$\mu(G) > k/2 + \sqrt{kn + (k^2 - 4k)/4}, \quad (10)$$

then $C_{2l+2} \subset G$ for every $l = 1, \dots, k$.

Theorem 3, together with (8) and (9), implies that

$$(k-1)/2 + \sqrt{kn} + o(n) \leq g_{2k+1}(n) \leq f_{2k+2}(n) \leq k/2 + \sqrt{kn} + o(n).$$

Finally, to determine the asymptotics of $g_{2k}(n)$ we need the following theorem.

Theorem 4. Let $k \geq 1$ and G be a graph of order n . If

$$\mu(G) > (k-1)/2 + \sqrt{kn + (k+1)^2/4},$$

then $C_{2k+1} \subset G$ or $C_{2k+2} \subset G$.

Theorem 4, together with (7), implies that for every $k \geq 1$,

$$g_{2k}(n) = (k-1)/2 + \sqrt{kn} + \Theta(n^{-1/2}).$$

3. Proofs

In this section we prove Theorems 2, 3 and 4. Unfortunately, the proofs are involved and require a number of known facts and some preliminary work.

3.1. Some known facts

We start with an upper bound on $\mu(G)$ that is particularly efficient for our problems.

Fact 5 [11]. If G is a graph with n vertices, m edges and $\delta(G) = \delta$, then

$$\mu(G) \leq (\delta-1)/2 + \sqrt{2m - \delta n + (\delta+1)^2/4}. \quad (11)$$

Note that for connected graphs inequality (11) has been proved independently by Hong et al. [9]. A particular instance of (11) is the following upper bound (see [15]): for every graph G with m edges,

$$\mu(G) \leq -1/2 + \sqrt{2m + 1/4} \leq \sqrt{2m}. \quad (12)$$

Next we turn to two classical results in extremal graph theory.

Fact 6 [5, Theorem 2.6]. Let $l \geq 2$ and G be a graph of order n . If $e(G) > (l/2)n$, then G contains a P_{l+2} .

Considerable work has been done to improve the above result under stronger assumptions, see, e.g. [1,6,10]. In particular, we shall use the following two facts, taken from [1].

Fact 7. Let $k \geq 1$, $n > 3k$ and G be a connected graph of order n . If

$$e(G) \geq e(S_{n,k}) = kn - (k^2 + k)/2, \quad (13)$$

then G contains a P_{2k+2} , unless there is equality in (13) and $G = S_{n,k}$.

Fact 8. Let $k \geq 1$, $n > 3k$ and G be a connected graph of order n . If

$$e(G) \geq e(S_{n,k}^+) = kn - (k^2 + k)/2 + 1, \quad (14)$$

then G contains a P_{2k+3} , unless there is equality in (14) and $G = S_{n,k}^+$.

Recently in [14], Fact 6 has been extended as follows:

Fact 9. Suppose that $k \geq 1$ and let the vertices of a graph G be partitioned into two sets U and W .

(A) If

$$2e_G(U) + e_G(U, W) > (2k - 2)|U| + k|W|, \quad (15)$$

then there exists a path of order $2k$ or $2k + 1$ with both ends in U .

(B) If

$$2e_G(U) + e_G(U, W) > (2k - 1)|U| + k|W|, \quad (16)$$

then there exists a path of order $2k + 1$ with both ends in U .

3.2. Preliminary lemmas supporting the proof of Theorem 2

The proof of Theorems 2 is based on Lemmas 10 through 14 below.

Lemma 10. Let G be a graph of order n with minimum degree $\delta(G) = \delta$ and $\mu(G) = \mu$. If (x_1, \dots, x_n) is a unit eigenvector to μ , then

$$\min\{x_1, \dots, x_n\} \leq \sqrt{\frac{\delta}{\mu^2 + \delta n - \delta^2}}.$$

Proof. Set $\sigma = \min\{x_1, \dots, x_n\}$. If $\sigma = 0$, the assertion holds trivially, so let assume that $\sigma > 0$. This implies also that $\delta > 0$. Selecting $u \in V(G)$ to satisfy $d_G(u) = \delta$, we have

$$\begin{aligned} \mu^2 \sigma^2 &\leq \mu^2 x_u^2 = \left(\sum_{i \in \Gamma(u)} x_i \right)^2 \leq \delta \sum_{i \in \Gamma(u)} x_i^2 \leq \delta \left(1 - \sum_{i \in V(G) \setminus \Gamma(u)} x_i^2 \right) \\ &\leq \delta (1 - (n - \delta) \sigma^2) = \delta - (\delta n - \delta^2) \sigma^2, \end{aligned}$$

implying that $(\mu^2 + \delta n - \delta^2) \sigma^2 \leq \delta$. The desired inequality follows. \square

Lemma 11. Let G be a graph of order n and let (x_1, \dots, x_n) be a unit eigenvector to $\mu(G)$. If u is a vertex satisfying $x_u = \min\{x_1, \dots, x_n\}$, then

$$\mu(G - u) \geq \mu(G) \frac{1 - 2x_u^2}{1 - x_u^2}.$$

Proof. Setting for short $\mu = \mu(G)$, we have

$$\mu x_u = \sum_{v \in \Gamma(u)} x_v \quad \text{and} \quad \mu = 2 \sum_{vw \in E(G)} x_v x_w.$$

Since Rayleigh's principle implies that

$$2 \sum_{vw \in E(G-u)} x_v x_w \leq \mu(G-u) \sum_{v \in V(G) \setminus \{u\}} x_v^2 = \mu(G-u) (1 - x_u^2),$$

we see that

$$\begin{aligned} \mu &= 2 \sum_{vw \in E(G-u)} x_v x_w + 2x_u \sum_{v \in \Gamma(u)} x_v = 2 \sum_{vw \in E(G-u)} x_v x_w + 2x_u^2 \mu \\ &\leq \mu(G-u) (1 - x_u^2) + 2x_u^2 \mu, \end{aligned}$$

and so,

$$\mu(G-u) \geq \mu \frac{1 - 2x_u^2}{1 - x_u^2},$$

as required. \square

Lemma 12. Let G be a graph of order n , let $\mu(G) = \mu$ and (x_1, \dots, x_n) be a unit eigenvector to μ . If u is a vertex satisfying $x_u = \min\{x_1, \dots, x_n\}$, then

$$\mu(G-u) \geq \mu \left(1 - \frac{1}{\mu^2/\delta + n - \delta - 1} \right).$$

Proof. Lemma 11 implies that

$$\mu(G-u) \geq \mu \frac{1 - 2x_u^2}{1 - x_u^2} = \mu \left(1 - \frac{x_u^2}{1 - x_u^2} \right).$$

On the other hand, by Lemma 10 we have

$$x_u^2 \leq \frac{\delta}{\mu^2 + \delta n - \delta^2}$$

and so

$$\begin{aligned} \mu(G-u) &\geq \mu \left(1 - \frac{\delta}{\mu^2 + \delta n - \delta^2 - \delta} \right) \\ &= \mu \left(1 - \frac{1}{\mu^2/\delta + n - \delta - 1} \right) \end{aligned}$$

completing the proof. \square

Lemma 13. Let the numbers a, k, n, s satisfy

$$k \geq 2, s \geq 1, \text{ and } n - s \geq 4k^3 + 4|a|(k-1).$$

Let the sequence x_0, \dots, x_s satisfy

$$x_0 \geq (k-1)/2 + \sqrt{kn-a}$$

and

$$x_{i+1} \geq x_i \left(1 - \frac{1}{x_i^2/(k-1) + n - i - k} \right)$$

for $0 \leq i < s$. Then, for every $i = 1, \dots, s$, we have

$$x_i \geq (k-1)/2 + \sqrt{k(n-i) - a + 1/2}.$$

Proof. Clearly it is enough to prove the assertion for $i = 1$ since it will follow by induction for all $i = 1, \dots, s$. Assume for a contradiction that

$$x_1 \leq (k-1)/2 + \sqrt{k(n-1) - a + 1/2}$$

and for short set

$$b = (k-1)/2 + \sqrt{kn - a}.$$

Since the function

$$x \left(1 - \frac{1}{x^2/(k-1) + n - k} \right)$$

is increasing in x , we have

$$\begin{aligned} (k-1)/2 + \sqrt{k(n-1) - a + 1/2} &\geq x_1 \\ &\geq x_0 \left(1 - \frac{1}{x_0^2/(k-1) + n - k} \right) \\ &\geq b \left(1 - \frac{1}{\mu^2/(k-1) + n - k} \right) \end{aligned}$$

and so,

$$\begin{aligned} \frac{1}{b^2/(k-1) + n - k} &\geq \frac{b - ((k-1)/2 + \sqrt{k(n-1) - a + 1/2})}{b} \\ &= \frac{kn - a - (k(n-1) - a + 1/2)}{b(\sqrt{kn - a} + \sqrt{k(n-1) - a + 1/2})} \\ &= \frac{k - 1/2}{b(\sqrt{kn - a} + \sqrt{k(n-1) - a + 1/2})} \\ &> \frac{k - 1/2}{2b\sqrt{kn - a}}. \end{aligned}$$

Hence, by the AM–GM inequality,

$$\frac{\sqrt{kn - a}}{k - 1/2} > \frac{1}{2} \left(\frac{b}{k-1} + \frac{n-k}{b} \right) \geq \sqrt{\frac{n-k}{k-1}}.$$

Squaring both sides of this inequality, we obtain

$$\begin{aligned} n - k &< (k-1) \frac{kn - a}{(k-1/2)^2} = \frac{(k^2 - k)n}{k^2 - k + 1/4} - \frac{a(k-1)}{(k-1/2)^2} \\ &= n - \frac{n}{4(k^2 - k + 1/4)} - \frac{a(k-1)}{(k-1/2)^2} \end{aligned}$$

and so,

$$n < 4(k^2 - k + 1/4)k - 4a(k-1) < 4k^3 - 4a(k-1),$$

a contradiction completing the proof. \square

Lemma 14. Let the numbers $c \geq 0$, $k \geq 2$, $n \geq 2^{4k}$, and let G be a graph of order n . If $\delta(G) \leq k - 1$ and

$$\mu(G) \geq (k - 1) / 2 + \sqrt{kn - k^2 + c},$$

then there exists a graph H satisfying one of the following conditions:

- (i) $\mu(H) > \sqrt{(2k + 1)|H|}$;
- (ii) $|H| \geq \sqrt{n}$, $\delta(H) \geq k$ and

$$\mu(H) > (k - 1) / 2 + \sqrt{k|H| - k^2 + c + 1/2}.$$

Proof. Using the following procedure, define a sequence of graphs G_0, \dots, G_k , satisfying $|G_i| = n - i$ for $i = 0, \dots, k$:

begin

set $G_0 = G$;

set $r = 0$;

while $\mu(G_r) \leq \sqrt{(2k + 1)|G_r|}$ and $\delta(G_r) \leq k - 1$ **do**

begin

select a unit eigenvector (x_1, \dots, x_{n-r}) to $\mu(G_r)$;

select a vertex $u_r \in V(G_r)$ such that $x_{u_r} = \min\{x_1, \dots, x_{n-r}\}$;

set $G_{r+1} = G_r - u_r$;

add 1 to r ;

end;

end.

Let $s = \min\{r, n - \lfloor \sqrt{n} \rfloor\}$. Note that for every $1 \leq i < s$, in view of $\delta(G_i) \leq k - 1$, Corollary 12 implies that

$$\mu(G_{i+1}) \geq \mu(G_i) \left(1 - \frac{1}{\mu^2(G_i) / (k - 1) + n - i - k}\right).$$

We shall prove that for every $i = 1, \dots, s$,

$$\mu(G_i) \geq (k - 1) / 2 + \sqrt{k(n - i) - k^2 + c + 1/2}. \quad (17)$$

Indeed, let $x_i = \mu(G_i)$ for $i = 0, \dots, s - 1$. Set $a = k^2 - c$ and note that

$$|G_i| \geq \lfloor \sqrt{n} \rfloor \geq 2^{2k} \geq 5k^3 + |c| k \geq 4k^3 + |k^2 - c| k \geq 4k^3 + |a| k.$$

With this selection of $a, k, s, n, x_0, \dots, x_s$, Lemma 13 implies inequality (17).

Furthermore, for every $1 \leq i < s$, inequality (17) implies that

$$\mu^2(G_i) > k(n - i - k)$$

and so, we find that

$$\begin{aligned} \mu(G_{i+1}) &\geq \mu(G_i) \left(1 - \frac{1}{\mu^2(G_i) / (k - 1) + n - i - k}\right) \\ &> \mu(G_i) \left(1 - \frac{1}{k(n - i - k) / (k - 1) + n - i - k}\right) \\ &= \mu(G_i) \left(1 - \frac{k - 1}{(2k - 1)(n - i - k)}\right). \end{aligned}$$

On the other hand, Bernoulli's inequality gives

$$\left(1 - \alpha \frac{1}{i + 1}\right) \geq \frac{i^\alpha}{(i + 1)^\alpha},$$

whenever $0 < \alpha < 1$ and $i > 0$. In particular, we see that

$$\left(1 - \frac{k-1}{(2k-1)(n-i-k)}\right) \geq (n-i+1-k)^{(k-1)/(2k-1)} (n-i-k)^{-(k-1)/(2k-1)},$$

and so, if $0 \leq i < s$, then

$$\mu(G_{i+1})(n-i+1-k)^{-(k-1)/(2k-1)} \geq \mu(G_i)(n-i-k)^{-(k-1)/(2k-1)}.$$

Taking the first and last terms of this chain of inequalities, and setting $H = G_s$, $p = |H|$, we find that

$$\begin{aligned} \mu(H)(p-k)^{-(k-1)/(2k-1)} &\geq \mu(G)(n-k)^{-(k-1)/(2k-1)} \\ &> \sqrt{k}(n-k)^{1/(4k-2)} \end{aligned}$$

and so,

$$\begin{aligned} \mu(H) &\geq \sqrt{k}(n-k)^{1/(4k-2)} p^{(k-1)/(2k-1)} \\ &= \sqrt{kp}(n-k)^{1/(4k-2)} p^{-1/(4k-2)}. \end{aligned}$$

If $p = \lfloor \sqrt{n} \rfloor$, we see that $p \leq 2\sqrt{n-k}$, and so,

$$\begin{aligned} \mu(H) &\geq \sqrt{kp}(n-k)^{1/(4k-2)} p^{-1/(4k-2)} \\ &\geq \sqrt{kp}(n-k)^{1/(8k-4)} 2^{-1/(4k-2)} > \sqrt{(2k+1)p}; \end{aligned}$$

thus H satisfies condition (i), completing the proof if $p = \lfloor \sqrt{n} \rfloor$.

Otherwise we have $p > \sqrt{n}$ and $\delta(H) \geq k$. In view of (17), H satisfies condition (ii), completing the proof of Lemma 14. \square

Proof of Theorem 2 for $k = 1$. For technical purposes we prove the case $k = 1$ of Theorem 2 separately. More precisely, we shall prove that if G is a graph of order n , then:

- (i) if $n > 5$ and $\mu(G) \geq \sqrt{n-1}$, then G contains a P_4 , unless $G = S_{n,1}$.
- (ii) if $n \geq 10$ and $\mu(G) \geq \mu(S_{n,1}^+)$, then G contains a P_5 , unless $G = S_{n,1}^+$.

Proof of part (i). First we prove that every connected graph G of order at least 4 satisfying $\mu(G) > \sqrt{n-1}$ contains a P_4 . Indeed, this is obvious if G contains a C_4 . If $C_4 \not\subseteq G$, the result of Favaron, Mahéo, and Saclé mentioned in the Introduction implies that G contains a triangle, say with vertices u, v, w . Since G is connected, there is an edge between the sets $\{u, v, w\}$ and $V(G) \setminus \{u, v, w\}$, so we get a P_4 .

Suppose that G is connected, $\mu(G) = \sqrt{n-1}$, and G contains no P_4 . Then G contains no cycles and so it must be a star $S_{n,1}$.

Suppose now that $n > 5$ and G is not connected. Take a component H with $\mu(H) = \mu(G) \geq \sqrt{n-1} > \sqrt{|H|-1}$. According to the argument for connected graphs, H must be of order 3 or 2. Thus, we have $2 \geq \sqrt{n-1}$, a contradiction.

Proof of part (ii). First we shall prove that every connected graph G of order at least 5 contains a P_5 unless G has no cycles or $G = S_{n,1}^+$. Indeed, every cycle longer than 4 contains P_5 , so we can assume that G contains no such cycles.

If G contains a C_4 , say with vertices u, v, w, t , there is an edge between the sets $\{u, v, w, t\}$ and $V(G) \setminus \{u, v, w, t\}$, so we get a P_5 . Thus, we can assume that G contains no C_4 .

Suppose that G contains a triangle, say with vertices u, v, w . If two of the vertices $\{u, v, w\}$ are joined to vertices from $V(G) \setminus \{u, v, w\}$, we get a P_5 . Hence only one of the vertices $\{u, v, w\}$ is joined to vertices belonging to $V(G) \setminus \{u, v, w\}$; let this be the vertex u . Since all vertices $V(G) \setminus \{u, v, w\}$ are joined by

some path to u , to avoid a P_5 , all vertices $V(G) \setminus \{u, v, w\}$ must be joined to u by an edge and the set $V(G) \setminus \{u, v, w\}$ must be independent. Therefore $G = S_{n,1}^+$.

Since a graph G with no cycles satisfies $\mu(G) \leq \sqrt{n-1} < \mu(S_{n,1}^+)$, part (ii) is proved for connected graphs of order at least 5.

Let now G be disconnected and $n \geq 10$. Take a component H with $\mu(H) = \mu(G) \geq \mu(S_{n,1}^+) > \mu(S_{|H|,1}^+)$. According to the argument for connected graphs, if H contains no P_5 , we have $|H| \leq 4$. But this is impossible since $\mu(H) = \mu(G) > \sqrt{n-1} \geq 3$.

Proof of Theorem 2 for $k \geq 2$. Clearly we can assume that G is connected. For short set $m = e(G)$.

Proof of part (a). If $\delta(G) \geq k$, inequality (11) implies that

$$\begin{aligned} \mu(G) &\leq (\delta - 1)/2 + \sqrt{2m - \delta n + (\delta + 1)^2/4} \\ &\leq (k - 1)/2 + \sqrt{2m - kn + (k + 1)^2/4}. \end{aligned}$$

Hence, in view of

$$\mu(G) \geq \mu(S_{n,k}) = (k - 1)/2 + \sqrt{kn - k^2 + (k - 1)^2/4},$$

we obtain

$$2m \geq 2kn - (k^2 + k) = 2e(S_{n,k}).$$

Now Fact 7 implies that G contains a P_{2k+2} unless $G = P_{2k+2}$. This completes the proof of part (a) if $\delta \geq k$.

Assume now that $\delta(G) \leq k - 1$. Applying Lemma 14 with $c = (k - 1)^2/4$, we find a graph H such that either (i) $\mu(H) > \sqrt{(2k + 1)|H|}$ or (ii) $|H| > \sqrt{n}$, $\delta(H) \geq k$ and

$$\mu(H) > (k - 1)/2 + \sqrt{k|H| - k^2 + (k - 1)^2/4}.$$

If (i) holds, then in view of (12), we see that

$$2e(H) \geq \mu^2(H) > (2k + 1)|H|$$

and so, by Fact 6, G contains a P_{2k+3} , completing the proof of part (a) in this case.

If (ii) holds, then we have

$$\mu(H) > (k - 1)/2 + \sqrt{k|H| - k^2 + (k - 1)^2/4}$$

and having from (11)

$$\mu(H) \leq (k - 1)/2 + \sqrt{2e(H) - k|H| + (k - 1)^2/4},$$

we find that

$$2e(H) > 2kp - (k^2 + k) = 2e(S_{n,k}).$$

Fact 7 implies that G contains a P_{2k+2} , completing the proof of part (a).

Proof of part (b). The proof goes like in Part (a), but needs more care. If

$$m \geq kn - (k^2 + k)/2 + 1,$$

the assertion follows from Fact 8, so we shall assume that

$$2m \leq 2kn - (k^2 + k).$$

If $\delta(G) \geq k$, inequality (11) gives

$$\begin{aligned}
 \mu(G) &\leq (\delta - 1)/2 + \sqrt{2m - \delta n + (\delta + 1)^2/4} \\
 &\leq (k - 1)/2 + \sqrt{2m - kn + (k + 1)^2/4} \\
 &= \mu(S_{n,k}) < \mu(S_{n,k}^+),
 \end{aligned}$$

a contradiction. Thus we have $\delta(G) \leq k - 1$.

Applying Lemma 14 with $c = (k - 1)^2/4$, we find a graph H such that either (i) $\mu(H) > \sqrt{(2k + 1)|H|}$ or (ii) $|H| > \sqrt{n}$, $\delta(H) \geq k$ and

$$\mu(H) > (k - 1)/2 + \sqrt{k|H| - k^2 + (k - 1)^2/4 + 1/2}.$$

If (i) holds, then in view of (12), we see that

$$2e(H) \geq \mu^2(H) > (2k + 1)p$$

and so, by Fact 6, G contains a P_{2k+2} .

If (ii) holds, then

$$\begin{aligned}
 \mu(H) &> (k - 1)/2 + \sqrt{k|H| - k^2 + (k - 1)^2/4 + 1/2} \\
 &= (k - 1)/2 + \sqrt{k|H| - (3k^2 + 2k - 3)/4}.
 \end{aligned}$$

We shall show that

$$(k - 1)/2 + \sqrt{k|H| - (3k^2 + 2k - 3)/4} > \mu(S_{p,k}^+).$$

Indeed, assume for a contradiction that this inequality fails and set for short $p = |H|$. In view of (4), we see that

$$\begin{aligned}
 \mu(S_{p,k}^+) &< \mu(S_{p,k}) + \frac{1}{p - k - 2\sqrt{(p - k)/k}} \\
 &= (k - 1)/2 + \sqrt{kp - (3k^2 + 2k - 1)/4} + \frac{1}{p - k - 2\sqrt{(p - k)/k}}.
 \end{aligned}$$

Therefore,

$$\sqrt{kp - (3k^2 + 2k - 3)/4} - \sqrt{kp - (3k^2 + 2k - 1)/4} < \frac{1}{p - k - 2\sqrt{(p - k)/k}}$$

and so,

$$\begin{aligned}
 \frac{1}{p - k - 2\sqrt{(p - k)/k}} &> \frac{1/2}{\sqrt{kp - (3k^2 + 2k - 3)/4} + \sqrt{kp - (3k^2 + 2k - 1)/4}} \\
 &> \frac{1}{4\sqrt{kp}}.
 \end{aligned}$$

Since $p \geq \lceil \sqrt{n} \rceil \geq 2^{2k}$, the above inequality is a contradiction.

The proof of Theorem 2 is completed. \square

Proof of Theorem 3. Since the expression

$$k/2 + \sqrt{kn + (k^2 - 4k)/4}$$

is increasing in k , it is enough to prove the existence only of C_{2k+2} . Assume for a contradiction that $C_{2k+2} \not\subseteq G$.

Select $u \in V$; let

$$U = \Gamma_G(u), W = V \setminus (\Gamma_G(u) \cup \{u\})$$

and set $H = G - u$. Since G contains no C_{2k+2} , H contains no path of order $2k + 1$ whose ends belong to U . By part (B) of Fact 9, we have

$$\begin{aligned} 2e_H(U) + e_H(U, W) &\leq (2k - 1)|A| + k|B| = (2k - 1)d_G(h) + k(n - d_G(h) - 1) \\ &= (2k - 1)d_G(h) + k(n - 1). \end{aligned}$$

Hence, we see that

$$\begin{aligned} \sum_{v \in U} d_G(v) &= d_G(u) + \sum_{v \in U} d_H(v) = d_G(u) + 2e_G(U) + e_G(U, W) \\ &\leq d_G(u) + (k - 1)d_G(h) + k(n - 1) \\ &= kd_G(u) + k(n - 1). \end{aligned}$$

Letting A be the adjacency matrix of G , note that the u 'th row sum of the matrix

$$A' = A^2 - kA$$

is equal to

$$\sum_{v \in \Gamma(u)} d_G(v) - kd_G(u),$$

consequently, the maximum row sum r_{\max} of C satisfies

$$r_{\max} \leq k(n - 1).$$

Letting \mathbf{x} be an eigenvector of A to μ , we see that the value

$$\lambda = \mu^2 - k\mu$$

is an eigenvalue of A' with eigenvector \mathbf{x} . Therefore,

$$\mu^2 - k\mu = \lambda \leq k(n - 1),$$

and so,

$$\mu \leq k/2 + \sqrt{kn + (k^2 - 4k)/4}.$$

This contradiction with (10) completes the proof of Theorem 3. \square

Proof of Theorem 4. The proof of Theorem 4 is identical to the proof of Theorem 3, except that part (A) of Fact 9 is used instead of part (B). \square

4. Two conjectures

The following conjecture, if true, will give the exact values of the functions $f_{2l}(n)$ and $g_l(n)$ for all $l > 2$ and n sufficiently large.

Conjecture 15. Let $k \geq 2$ and G be a graph of sufficiently large order n .

- (a) if $\mu(G) \geq \mu(S_{n,k})$, then G contains C_{2k+1} or C_{2k+2} unless $G = S_{n,k}$;
- (b) if $\mu(G) \geq \mu(S_{n,k}^+)$, then G contains C_{2k+2} unless $G = S_{n,k}^+$.

We finish with a conjecture that goes beyond cycles and paths. It is motivated by the famous Erdős-Sós conjecture about the maximum number of edges in a graph of order n that does not contain some tree of order k .

Conjecture 16. Let $k \geq 2$ and G be a graph of sufficiently large order n .

- (a) if $\mu(G) \geq \mu(S_{n,k})$, then G contains all trees of order $2k + 2$ unless $G = S_{n,k}$;
 (b) if $\mu(G) \geq \mu(S_{n,k}^+)$, then G contains all trees of order $2k + 3$ unless $G = S_{n,k}^+$.

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